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# Mathematical Reflective Thinking Processesof Senior High School Students

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## ABSTRACT

Reflective thinking is an important aspect that should be developed by students because they often face problems which are not immediately able to be solved, while students are required to be able to resolve them. Students need to have the awareness to think, predict, and seek simple formulas, and then prove the truth. This research aims to describe the processes of reflective thinking of tenth-grade students in solving mathematical problems. This research is qualitative in nature. Subjects are students in the tenth-grade of MAN (Madrasah Aliyah Negeri) Ngawi in the second semester of the 2015/2016 academic year. Data were collected using task-based interviews and are analysed using Miles and Huberman's technique (1994) consisting of three activities; data reduction, data presentation, and conclusion. Results showed that students with high and average initial mathematical ability were already able to apply reflective thinking processes in solving mathematical problems. The results of this research indicated that students with low initial capability in Mathematics did not use reflective thinking processes for problem solving. Therefore, teachers needed to provide scaffolding to them so that they could apply reflective thinking in solving mathematical problems.

*Keywords:* Initial mathematical ability, problem solving, reflective thinking process

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## INTRODUCTION

Mathematics represents a human life activity, i.e. the problem-solving activity. Problem solving is a process of thinking (Krulik & Rudnick, 1988, p. 3). In solving problems, students are faced with an unusual situation and required to apply the knowledge, skills, and understanding which have been acquired to find and analyse a solution. The problems presented are aimed to make students think systematically of possible ways and results obtained, organise a range of knowledge and experiences, as well as save and recall a wide range of knowledge and skills to facilitate the process of problem solving (NCTM, 2015). Problems have actually been solved if the students understand what they are doing – that is, to understand the problem-solving process and understand why the obtained solution is appropriate (McIntosh, 2000). When a student observes, asks questions, and understands the meaning of this problem - that is when the student has a good opportunity to think reflectively. Reflection involves the critique of assumptions about content or the process of problem solving (YuekMing & Manaf, 2014). Thus, reflection helps students to develop higher-order thinking skills by prompting them to: (a) relate new knowledge to their prior understanding, (b) think in both abstract and concrete terms, (c) apply specific strategies to novel tasks, and (d) understand their own thinking and learning strategies (Hmelo & Ferrari, 1997). Gagatsis and Patronis (1990) refer to it as the initial process of reflective mathematical thinking. Dewey (1933) suggests that reflective thinking is an active, persistent, and careful consideration of a belief or supposed form of knowledge, of the grounds that support that knowledge, and the further conclusion to which that knowledge leads.

The process of reflective mathematical thinking begins to develop at the age of 7 years old. At that age, a child is able to manipulate a variety of concrete ideas and recount what has been done (in his imagination) (Skemp, 1982). This is reinforced by Gagatsis and Patronis' research (1990) which found that after the age of 7–8 years old, the process of reflective thinking is relatively stable, especially in the determination of problem-solving strategies (geometry model). Furthermore, Gagatsis and Patronis recommended conducting advanced research related to the process of reflective thinking among students in more mature age; this is because the reflective thinking is an important aspect that should be developed by students in learning (Ayazgok & Aslan, 2004; Odiba & Baba, 2013). Reflective thinking can be used as a means of encouraging the process of thinking during problem solving because it gives students the opportunity to predict the correct answers promptly so as to explore the problem by identifying the mathematical concepts involved, using a variety of strategies, building ideas, drawing conclusions, determining the validity of the argument, re-examining solutions, and developing alternative strategies (Kurniawati, Kusumah, Sumarmo, & Sabandar, 2014). To be able to pass any of these activities, of course, it requires the involvement of mental activity.

The interactions that occur when students respond to the external environment, followed by intervening mental activity, are called reflective thinking processes

(Skemp, 1982). The intervening mental activity becomes the conscious object of self-introspective (introspective awareness) that generates responses (effectors). This is confirmed by the results of Kosslyn's research (2005) which showed that the process of reflective thinking occurs when information stored in the long-term memory (LTM) does not allow a person to respond automatically to an object or event. The stored information should be included in the working memory (WM) and then a new response or solution is produced. Furthermore, mental imagery plays an important role in the process of reflective thinking. An understanding of mental imagery provides information related to the cognitive developmental disorders such as the failure of a student to apply reflective thinking in the right condition even when he/she cannot do any processing information effectively. Dewey (1933) stated that the knowledge and experience previously gained by students would affect the process of reflective thinking, while Hamdi (2012) reported that mathematical abilities essentially affect students' reflective thinking processes. The variation degree of the answers which has accuracy scope and requires the development of existing formulas will facilitate students to find solutions. This is in line with Moon (1999) who stated that the process of reflective thinking among students is different and evolving with age. Galton (Ibrahim, 2011) emphasised that among a group of students selected randomly, there would always be those with high, medium, and low abilities.

On the other hand, the process of mathematical problem solving is related to the stages of problem solving being carried out (Usodo, 2012). Although students master the steps to resolve the problem, they may sometimes find difficulties in solving the problem. In relation to the process of reflective thinking in problem solving, the reflective thinking process through which students are in the problem-solving process can be tracked down from the stages of problem solving. Based on relevant theories and research results, the researcher collected data from two students so that they have different initial mathematical abilities to reveal the process of reflective thinking used in problem solving. Results of this analysis on the data with diverse students' ability, there is a need for further research which aims to describe the process of reflective mathematical thinking among students of Madrasah Aliyah who possess high, medium, and low initial mathematical abilities in solving the problem based on Krulik and Rudnick's (1988) steps.

## METHODS

The present study is a qualitative research which aims to analyse the mathematical reflective thinking processes of senior high school students which fulfil credibility, reliability and reveal their characteristics. Data collection was conducted by means of task-based interviews. The research was conducted in MAN (Madrasah Aliyah Negeri) Ngawi, in the second semester of the academic year 2015/2016. The range of students' age was between fifteen and seventeen years old. The subjects were students in the tenth grade of MAN Ngawi who had completed trigonometry subject. Selection of the research subject was done via purposive sampling technique. The main instruments were the researchers themselves as interviewers assisted by forms of testing, solutions and interview guidelines. Validity of the data was done by triangulating time and increasing endurance. The data in the research were the reflective thinking processes of students who have high, medium, and low initial mathematical abilities in problem solving. The sources of the data were the work of the subjects, interviews with the subjects, and field notes, while the data collection techniques were task-based interviews. Miles and Huberman's data analysis techniques were applied in the research.

The research involved twelve tenth-graders of MAN Ngawi under the following research procedures: (a) formulating the early hypothetic theory based on a literature review and supported by the early empiric data; (b) validating the draft for the mathematical reflective thinking processes to the experts to reveal the content validity and the theoretical construct to be developed; (c) revising the draft for the mathematical reflective thinking processes by proposing a new theory; (d) collecting data to reveal the existence of mathematical reflective thinking processes; (e) performing analysis using fixed comparison to reveal the dependability of mathematical reflective thinking processes, and (f) writing out the mathematical reflective thinking processes.

## RESULTS AND DISCUSSION

### The processes of mathematical reflective thinking ability of MAN students having high initial mathematical ability

During the reading and thinking step, students with high initial mathematical ability in solving mathematical problems believed that what they read and thought was right by reading the questions repeatedly, giving meaning to every sentence in solving problems, and representing problems. In the exploring and planning step, they selected and considered a variety of information to plan the problem solution by: (a) analysing the concept or the information on the subject matter and the problem situation; (b) generating and checking the truth of the information that will be used; and (c) using their intuition and asking themselves (self-questioning) to believe whether such information could be used to plan a resolution. The students believed that the original plan for problem solving was drawn up right by: (a) organising the problem; (b) deciding firmly on various prepared initial plans; and (c) using their intuition and asking themselves to believe that the plan drawn up is correct. Here are the results of the representation of the subject on the task of solving the problem. A circle is drawn touching the sides of an equilateral triangle. Inside the circle, there is a square whose four vertexes lie on the circle (Figure 1). How is the ratio of the square's width and the equilateral triangle's?

*Figure 1*. The process of problem representations of students having high initial mathematical ability

Solso, Maclin, and Maclin (2008, p. 297) call the visual representation of knowledge a parable or mental imagery. Kaldrimidou (as cited in Gagatsis & Patronis, 1990) noted that mental imagery is a mental object to do the process of reflective thinking. In addition, the mental imagery will help students to solve problems, reorganise the range of knowledge and reconstruct the concept of the problem. Clark and Paivio (Solso et al., 2008, p. 311) stated that the LTM has two means to represent knowledge, namely, the verbal system and imaging system. The verbal system incorporates knowledge expressed through language and imaging system, or image system, as well as stores visual and spatial information. Both systems are interrelated; a verbal code can be converted into an imaginary code or image code, and vice versa. Unlike the two-code theory, the unitary theory states that all information is represented in the LTM in the form of verbal codes (propositions).

In determining strategy, students considered the problem-solving strategy which was determined based on data and information. The data and information were obtained by (a) developing an initial plan for the completion of the work in the representation results in a trial-error and guess-test; (b) determining the pattern of problem solving; (c) checking each process; and (d) using their intuition and asking themselves to believe that the chosen problem-solving strategy was correct. Feldman (2012, p. 314) stated that finding a solution can be done by using a trial-error strategy. The completion of complex problems often involves the use of heuristics, short cognitive which can produce a solution. The most often used heuristics in problem solving is means-end analysis. A means-end analysis of every step leads students closer to a resolution. While this approach is often effective if the problem requires indirect measures which temporarily increase the discrepancy between a current state and the solution, the means-end analysis becomes unproductive.

In determining completion, students understood every step of the work based on problem-solving strategies which were selected by (a) ensuring the applied formula (area of plane, area of a triangle if it is known that two angles are adjoining a corner, and a comparison of trigonometry to specific angles based on firm and accurate considerations); (b) doing repeatedly using the selected pattern; (c) checking and observing each step of the work and calculations by working backwards; (d) being aware of the errors (in the formula, computing, and writing) and improving them; and (e) using their intuition and asking themselves to believe the steps of the work according to the chosen method. Some errors made by the subjects are shown in Figure 2 below:

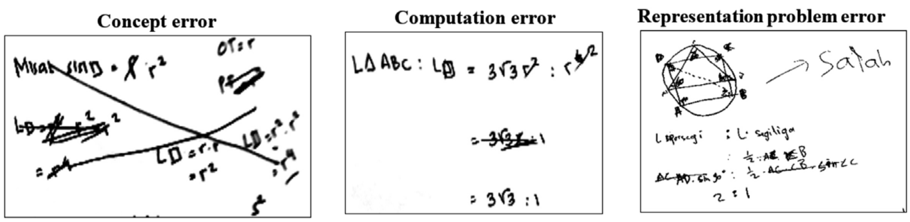


*Figure 2*. Errors made by students having high initial mathematical ability

In reflecting and generalising, students considered the appropriateness of the obtained results for the existing problems by (a) reflecting on each process used to obtain a solution; (b) testing the correctness of the conclusion drawn by verifying information; and (c) using their intuition and asking themselves to believe that the obtained solution had answered the problems.

One interesting thing shown by the subjects having high initial mathematical ability in any problem-solving steps is that they always asked themselves questions (self-questioning). This result of the research confirms the research carried out by Teekman (2000). Teekman noted that asking ourselves questions is important in supporting the development of reflective thinking processes. Furthermore, asking ourselves questions is used to create and understand the meaning of a mental object. Next, some benefits to asking ourselves questions are: (a) helping students to clarify and categorise situations and events, and contributing to logical thinking skills; (b) helping to structure the thinking process and reducing the possibility of overlooking the important aspects of an event; (c) helping students to make sense of a situation and planning what to do next; and (d) helping students to clarify existing schemes or rules which they use to respond within a broader context.

Another thing done by the students having high initial mathematical ability is using their intuition to convince them that what they have done is right. Intuition is described by Fischbein (1999) as a cognition which subjectively retains truth in it, may be accepted by itself, and is direct, holistic, leading, and estimating. One of the intuition characteristics stated by Fischbein (1999) is coerciveness, i.e. an intuition which naturally leads towards a believed thing. This indicates that students tend to reject an alternative interpretation which would contradict their intuition. This is in line with Hogarth's research (2001) which states that intuition will be present and used when one deals with the dilemma of problem solving or decision making. The process underlying the intuition used in problem solving is matching the pattern which can be sharpened through continuous training and practice. In line with this, Fischbein (1999) remarks that intuition can be used as a mediating cognitive employed as a bridge of a person's understanding, so it can help and facilitate in linking the imaged object with a desired alternative solution.

### The reflective thinking processes of MAN students having average initial mathematical ability

During the step of reading and thinking, students with average initial mathematical ability believed that what they read and thought is right by reading the questions repeatedly and representing the problem. In the exploring and planning step, students selected and considered a variety of information for their problem-solving plans by analysing the concept or the information on the subject matter and problem situations. Students believed that the initial plan was proper by organising the problems and deciding on the prepared initial plan.

In choosing a strategy, the students considered problem solving plans/strategies which were determined based on the data and information obtained with (a) the development of an initial completion plan formed by working on the results of representation through trial and error; and (b) the decisions of solving problems with a pattern which was properly selected but with question stimuli.

In determining completion, the students understood that every step of the work, which was based on the chosen problem-solving strategy, was correct by (a) ensuring the formula used in relation to the plane area, the area of a triangle if it is known that two angles are adjoining a corner, and a proper trigonometry comparison of special angles, but that the questions should be assisted by stimuli; (b) doing repeatedly using the selected pattern and checking every step of work and calculations performed; and (c) being aware of the errors (in the completion strategy, formula, computing, and writing) and fixing them with the question stimuli. An interesting phenomenon is that students worked in two ways which resulted in different answers. In relation to this phenomenon, Feldman (2012, p. 315) explained that this occurs because of the sudden awareness (insight). Several approaches to creating possible solutions focus less on the step-by-step heuristics as compared to the sudden emergence of understanding that may be experienced by a person when they are trying to solve a problem. The phenomenon is caused by an inaccurate evaluation of the solution, which is hereinafter known as confirmation bias. Confirmation bias is a tendency to look for and be in favour of information that supports one's initial hypothesis and avoid the opposing information that supports the alternative hypothesis or solution. This occurs because the confirmation bias rethinks a problem that seems to have been resolved, and students tend to stick with the first solution. In the reflecting and generalising step, students consider the appropriateness of the results obtained for the existing problems by correctly checking and ensuring basic problems and obtained solutions.

### The reflective thinking processes of MAN students having low initial mathematical ability

The students with low mathematical ability did not use their reflective thinking processes in problem solving, as mentioned by Krulik and Rudnick (1988). However, it was noted that the students in this category often said "I forget" and were aware of the mistake but did not know how to fix it. They forgot to refer to the portion of the knowledge or their ability was lost in their memory. The forgetfulness symptom easily occurs in cognitive knowledge when students do not successfully construct their own knowledge or associate knowledge or concepts they have learned with the knowledge or concepts which they have had (2004, pp. 446–452). Furthermore, one of the causes of forgetting is that students do not get the right key to unlock their memory, so the difficulty arises in the phase of recalling itself. The forgetfulness symptom is due to a lack of attention to the concentration phase and to imperfect material management (fixation) before it is put into the LTM. Most of the materials which should be processed are touched briefly and then left out; most of the materials are processed in the short-term memory (STM) imperfectly and then put into the LTM in a state of half-understanding. As a result, there is no good organisation in the LTM. Meanwhile, psychologists have claimed four things which cause forgetfulness; they are the failure to retrieve, reconstruction errors, interference, and decay (Ormrod, 2008).

The interview results showed that the subject was aware of the error, but he/she was unable to correct it. An example of the error made by the student when he/she described it (Figure 3).

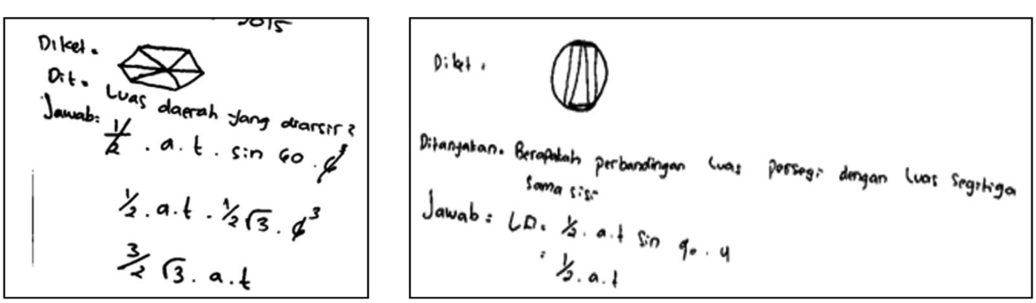


*Figure 3*. Errors made by students having low initial mathematical ability

Based on the chart, Skemp (1982, pp. 54-55) described the phenomenon of when students realise their error because during the mental activity (intervening mental activities) awareness for introspection (introspective awareness) occurs. However, students are not able to find a solution to the introspection.

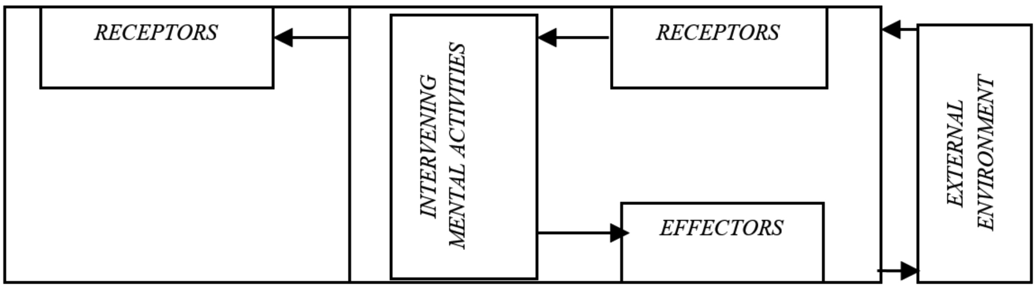


*Figure 4*. The reflective thinking process

## CONCLUSION

Based on the results of the research and discussion, the conclusion is as follows. The processes of reflective thinking of MAN students having high initial mathematical ability in problem solving based on Krulik and Rudnick's (1988) steps are: (a) the students believe what they read and think is right by reading the questions repeatedly, giving meaning to each sentence, and representing the problem; (b) students select and consider a variety of information to make the initial plan of problem solving by analysing the concept or the information on the subject matter and the problem situation, and generating and checking the appropriateness of the information that will be used. The students believe that the initial plan of the problem solving is correct by organising the problem, and deciding firmly on various initial prepared plans; (c) students consider the problem-solving strategy, which is determined based on the data and information obtained by developing an initial plan for the completion of the work in the representation results in a trial-error and guess-test, determining the pattern of problem solving, and checking each process; (d) students understand every step of the work based on problem-solving strategies which are selected by ensuring the applied formula, doing repeatedly using the selected pattern, checking and observing each step of work and the calculations by working backwards, being aware of the errors and improving them; and (e) in reflecting and generalising, students consider the appropriateness of the obtained results for the existing problems by reflecting on each process to obtain a solution, and testing the correctness of the conclusion drawn by verifying the information. In every step of problem solving, they always use their intuition and ask themselves to make sure that what has been done is right.

The reflective thinking processes of MAN students with average initial mathematical ability in solving problems based on Krulik and Rudnick's (1988) steps but not completely the high initial mathematical ability. Students with low mathematical ability do not use their reflective thinking processes in problem solving, as mentioned by Krulik and Rudnick. Therefore, teachers need to provide specific scaffolding for these students so that they are able to think reflectively and solve mathematical problems well.

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## REFERENCES

Ayazgok, B., & Aslan, H. (2014). The review of academic perception, level of metacognitive awareness and reflective thinking skills of science and mathematics university student. *Procedia - Social and Behavioral Sciences, 141*, 781–790.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process.* Boston, MA: D.C., Heath and Company. Retrieved from rci.rutgers.edu_-tripmcc_phil_dewey-hwt-pt1-selections.pdf

Feldman, R. S. (2012). *Understanding psychology* (10th ed.). New York: McGraw-Hill.

Fischbein, E. (1999). Intuitions and schemata in mathematical reasoning. *Educational Studies in Mathematics, 38*, 27–47.

Gagatsis, A., & Patronis, T. (1990). Using geometrical models in a process of reflective thinking in learning and teaching mathematics. *Educational Studies in Mathematics Netherlands, 21*, 29-54.

Gurol, A. (2011). Determining the reflective thinking skills of pre-service teachers in learning and teaching process. *Energy Education Science and Technology Part B: Social and Educational Studies, 3*(3), 387–402.

Hamdi, S. (2012). *Memahami karakteristik psikologis siswa dalam pembelajaran matematika berdasarkan kecerdasan intuitif dan reflektif [Understanding characteristic of student's psychological in learning mathematics based on reflective and intuitive intelligence]*. Paper presented in national seminar of mathematic and mathematic education entitle "The Contribution of Mathematic and Mathematic Education in Building Teacher and Student Character" 10th November 2012, ISBN: 978-979-16353-8-7. Yogyakarta: FMIPA UNY.

Hmelo, D., & Ferrari, M. (1997). The problem-based learning tutorial: Cultivating higher order thinking skills. *Journal for the Education of the Gifted, 20*(4), 401–422.

Hogarth, R. (2001). *Educating intuition.* Chicago: University of Chicago Press.

Ibrahim. (2011). *Peningkatan kemampuan komunikasi, penalaran, dan pemecahan masalah matematis serta kecerdasan emosional melalui pembelajaran berbasis masalah pada siswa sekolah menengah atas [Enhancement of communication ability, reasoning, mathematical problem solving and emotional intelligence through problem-based learning for student senior high school]* (Unpublished dissertation, University of Education, Indonesia).

Kosslyn, S. M. (2005). Reflective thinking and mental imagery: A perspective on the development of posttraumatic stress disorder. *Development and Psychopathology Journal, 17*, 851–863. Cambridge University Press.

Krulik, S., & Rudnick, J. A. (1988). *Problem solving: A handbook for teachers*. Boston: Allyn & Bacon.

Kurniawati, L., Kusumah, Y.S., Sumarmo, U., & Sabandar, J. (2014). Enhancing students' mathematical intuitive-reflective thinking ability through problem-based learning with hypoteaching method. *Journal of Education and Practice, 5*(36).

McIntosh, R. (2000). *Teaching mathematical problem solving: Implementing the visions*. Retrieved from http://www.nwrel.org/msec/images/mpm/pdf/monograph.pdf.

Miles, M. B. and Huberman, A. M. (1994). *Qualitative Data Analysis* (2nd ed.). Thousand Oaks, CA : Sage publications.

Moon, J. (1999). *Reflection in learning and professional development: Theory and practice.* London: Kogan Page.

National Council of Teachers of Mathematics (NCTM). (2015). *Classroom resources*. Retrieved from http://www.nctm.org/Conferences-and-Professional-Development/Research-Conference/

Odiba, I. A., & Baba, P. A. (2013). Using reflective thinking skills for education quality improvement in Nigeria. *Journal of Education and Practice, 4*(16).

Ormrod, E. J. (2008). *Psikologi pendidikan [Education psychology]*. Jakarta: Erlangga.

Skemp, R. R. (1982). *The psychology of learning mathematics.* Great Britain: Penguin Books.

Solso, L. R., Maclin, H. O., & Maclin, K. M. (2008). *Psikologi kognitif [Cognitive psychology]*. Jakarta: Erlangga.

Usudo, B. (2012). Karakteristik intuisi siswa MAN dalam memecahkan masalah matematika ditinjau dari kemampuan matematika dan perbedaan gender [Characteristic of student intuition in solving mathemnatic problems, based on mathematic competence and gender]. *AKSIOMA Jurnal Pendidikan Matematika UNTAD, 1*(1).

Winkel, W.S. (1996). *Psikologi pengajaran [Instructional psychology]*. Jakarta: Gramedia.

Yuek Ming, H., & Manaf, L. A. (2014). Assessing learning outcomes through students' reflective thinking. *Procedia - Social and Behavioral Sciences, 152*, 973–977.